The skew energy of random oriented graphs*

Xiaolin Chen, Xueliang Li, Huishu Lian
Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, P.R. China
E-mail: chxlnk@163.com; lxl@nankai.edu.cn; lhs6803@126.com

Abstract

Given a graph G, let G^σ be an oriented graph of G with the orientation σ and skew-adjacency matrix $S(G^\sigma)$. The skew energy of the oriented graph G^σ, denoted by $\mathcal{E}_S(G^\sigma)$, is defined as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$. In this paper, we study the skew energy of random oriented graphs and formulate an exact estimate of the skew energy for almost all oriented graphs by generalizing Wigner's semicircle law. Moreover, we consider the skew energy of random regular oriented graphs $G^\sigma_{n,d}$, and get an exact estimate of the skew energy for almost all regular oriented graphs.

Keywords: skew energy, random graph, oriented graph, random matrix, eigenvalues, empirical spectral distribution, limiting spectral distribution, moment method

AMS Subject Classification Numbers: 05C20, 05C80, 05C90, 15A18, 15A52

1 Introduction

Let G be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let G^σ be an oriented graph of G with the orientation σ, which assigns to each edge of G a direction so that the induced graph G^σ becomes a directed graph. The skew-adjacency matrix of G^σ is the $n \times n$ matrix $S(G^\sigma) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $\langle v_i, v_j\rangle$ is an arc of G^σ, otherwise $s_{ij} = s_{ji} = 0$. The skew energy [1] of G^σ is defined as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$, denoted by $\mathcal{E}_S(G^\sigma)$. Obviously, $S(G^\sigma)$ is a skew-symmetric matrix, and thus all the eigenvalues are purely imaginary numbers.

Since the concept of the energy of simple undirected graphs was introduced by Gutman in [8], there have been lots of research papers on this topic. We refer the survey [9] and the book [12] to the reader for details. The energy of a graph has a close link to chemistry. An important quantum-chemical characteristic of a conjugated molecule is its total π-electron energy. There

*Supported by NSFC and the "973" project.

are situations when chemists use digraphs rather than graphs. One such situation is when vertices represent distinct chemical species and arcs represent the direction in which a particular reaction takes place between the two corresponding species. It is reasonable to expect that the skew energy has similar applications as energy in chemistry.

Adiga et. al. [1] first defined the skew energy of an oriented graph, and obtained some properties of the skew energy. They derived an upper bound for the skew energy and constructed a family of infinitely many oriented graphs attaining the maximum. They also proved that the skew energy of an oriented tree is independent of its orientation, and equal to the energy of its underlying tree. Then, Shader et. al. [15] studied the relationship between the spectra of a graph G and the skew-spectra of an oriented graph G^σ of G, which would be helpful to the study of the relationship between the energy of G and the skew energy of G^σ. Hou and Lei [10] characterized the coefficients of the characteristic polynomial of the skew-adjacency matrix of an oriented graph. Moreover, other bounds and the extremal graphs of some classes of oriented graphs have been established. In [11] and [16], Hou et. al. determined the oriented unicyclic graphs with minimal and maximal skew energy and the oriented bicyclic graphs with minimal and maximal shew energy, respectively. Gong and Xu [7] characterized the 3-regular oriented graphs with optimum skew energy.

It is well known that it is rather hard to compute the eigenvalues for a large matrix and by the extremal graphs we can hardly see the major behavior of the invariant $\mathcal{E}_S(G)$ for most oriented graphs with respect to other graph parameters. Therefore, in this paper, we will study the skew energy in the setting of random oriented graphs. We first formulate an exact estimate of the skew energy for almost all oriented graphs by generalizing Wigner's semicircle law. Moreover, we investigate the skew energy of random regular oriented graphs, and also obtain an exact estimate.

Various energies of random graphs have been studied. Du et. al. considered the Laplacian energy in [4] and the energy in [6]. Moreover, they also investigated other energies in [5], such as the signless Laplacian energy, incidence energy, distance energy and the Laplacian-energy like invariant. It is worth to point out that their results depend on the limiting spectral distribution of a random real symmetric matrix. But our results on the skew energy of a random oriented graph relies on the limiting spectral distribution of a random complex Hermitian matrix.

The rest of the paper is organized as follows: In Section 2, we will list some notations and collect a few auxiliary results. Then in Section 3, we will consider the skew energy of random oriented graphs. Finally, in Section 4, we will be devoted to estimating the skew energy of random regular oriented graphs.

2 Preliminaries

In this section, we state some notations and collect a few results that will be used in the sequel of the paper.

Given an Hermitian matrix M on order n, denote its n eigenvalues by

$$\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M),$$

and the *empirical spectral distribution (ESD)* of the matrix M by

$$F_n^M(x) = \frac{1}{n}\Big|\{\lambda_i(M)|\lambda_i(M) \le x, i = 1, 2, \ldots, n\}\Big|$$

where $|I|$ means the cardinality of the set I. The distribution to which the ESD of the random matrix M converges as $n \to \infty$ is called the *limiting spectral distribution (LSD)* of M.

The study on the spectral distribution of random matrices plays a critical role in estimating the skew energy of random oriented graphs. One pioneer work in the field of the spectral distribution of random matrices [2,14] is Wigner's semicircle law discovered by Wigner in [18,19], which characterizes the limiting spectral distribution of a sort of random matrices. This sort of random matrices is so-called the *Wigner matrices*, denoted by X_n, which satisfies that

- X_n is a Hermitian matrix, i.e., $x_{ij} = \overline{x_{ji}}$, $1 \le i \le j \le n$, where $\overline{x_{ji}}$ means the conjugate of x_{ji},
- the upper-triangular entries x_{ij}, $1 \le i < j \le n$, are independently identically distributed (i.i.d.) complex random variables with mean zero and unit variance,
- the diagonal entries x_{ii}, $1 \le i \le n$, are i.i.d. real random variances, independent of the upper-triangular entries, with mean zero,
- for each positive integer k, $\max\big(\mathbf{E}(|x_{11}|^k), \mathbf{E}(|x_{12}|^k)\big) < \infty$.

Then the Wigner's semicircle law can be stated as follows.

Theorem 2.1 *[19] Let X_n be a Wigner matrix. Then the empirical spectral distribution $F_n^{n^{-1/2}X_n}(x)$ converges to the standard semicircle distribution whose density is given by*

$$\rho_{sc}(x) := \frac{1}{2\pi}\sqrt{4-x^2}\ _{|x|\le 2}.$$

Given a random graph model $\mathcal{G}(n,p)$, we say that *almost every* graph $G(n,p) \in \mathcal{G}(n,p)$ has a certain property $\mathcal{P}$ if the probability that $G(n,p)$ has the property $\mathcal{P}$ tends to 1 as $n \to \infty$, or we say $G(n,p)$ *almost surely (a.s.)* satisfies the property $\mathcal{P}$. In the sequel, we shall consider two random graph models: the *random oriented graph* model $\mathcal{G}^\sigma(n,p)$ and the *random regular oriented graph* model $\mathcal{G}^\sigma_{n,d}$, the definitions of which will be given later.

In this paper, we use the following standard asymptotic notations: as $n \to \infty$, $f(n) = o(g(n))$ means that $f(n)/g(n) \to 0$; $f(n) = \omega(g(n))$ means that $f(n)/g(n) \to \infty$; $f(n) = O(g(n))$ means that there exists a constant C such that $|f(n)| \leq Cg(n)$; $f(n) = \Omega(g(n))$ means that there exists a constant $c > 0$ such that $f(n) \geq cg(n)$.

3 The skew energy of $G^\sigma(n,p)$

In this section, we consider random oriented graphs and obtain an estimate of the skew energy for almost all oriented graphs by generalizing Wigner's semicircle law.

We first give the definition of a random oriented graph $G^\sigma(n,p)$. Given $p = p(n)$, $0 \leq p \leq 1$, a random oriented graph on n vertices is obtained by drawing an edge between each pair of vertices, randomly and independently, with probability p and then orienting each existing edge, randomly and independently, with probability $1/2$. That is to say, for a given oriented graph $G = G^\sigma(n,p)$ with m arcs, $P(G) = p^m(1-p)^{\binom{n}{2}-m} \cdot 2^{-m}$. Apparently, the skew-adjacency matrix $S(G^\sigma(n,p)) = [s_{ij}]$ (or S_n, for brevity) of $G^\sigma(n,p)$ is a random matrix such that

- S_n is skew-symmetric, i.e., $s_{ij} = -s_{ji}$ for $1 \leq i \leq j \leq n$, and in particular, $s_{ii} = 0$ for $1 \leq i \leq n$;
- the upper-triangular entries s_{ij}, $1 \leq i < j \leq n$ are i.i.d. random variables such that $s_{ij} = 1$ with probability $\frac{1}{2}p$, $s_{ij} = -1$ with probability $\frac{1}{2}p$, and $s_{ij} = 0$ with probability $1 - p$.

It is well known that all the eigenvalues of S_n are purely imaginary numbers. Assume that $i\lambda_1, i\lambda_2, \ldots, i\lambda_n$ are all the eigenvalues of S_n where every λ_k is a real number and i is the imaginary unit. Let $S'_n = (-i)S_n$. Then S'_n is an Hermitian matrix with eigenvalues exactly $\lambda_1, \lambda_2, \ldots, \lambda_n$. Therefore, the skew energy $\mathcal{E}_S(G^\sigma(n,p))$ can be evaluated once the spectral distribution of the random Hermitian matrix S'_n is known.

Usually, it is more convenience to study the normalized matrix $M_n = \frac{1}{\sqrt{p}} S'_n = [m_{ij}]$. Apparently, M_n is still an Hermitian matrix in which the diagonal entries $m_{ii} = 0$ and the upper-triangular entries m_{ij}, $1 \leq i < j \leq n$ are i.i.d. copies of random variable ξ which takes value $\frac{i}{\sqrt{p}}$ with probability $\frac{1}{2}p$, $-\frac{i}{\sqrt{p}}$ with probability $\frac{1}{2}p$, and 0 with probability $1-p$. It can be verified that the random variable ξ has mean 0, variance 1, and expectation

$$\mathbf{E}(\xi^s) = \begin{cases} 0 & \text{if } s \text{ is odd;} \\ \frac{1}{(\sqrt{p})^{s-2}} & \text{if } s \equiv 0 \mod 4; \\ -\frac{1}{(\sqrt{p})^{s-2}} & \text{if } s \equiv 2 \mod 4. \end{cases} \tag{3.1}$$

Observe that if $p = o(1)$, then the matrix M_n is not a Wigner matrix since the moment is unbounded as $n \to \infty$, and thus the limiting spectral distribution of M_n cannot be directly

derived by the Wigner's semicircle law. However, by the moment method, we can establish that the empirical spectral distribution of $\frac{1}{\sqrt{n}}M_n$ also converges to the standard semicircle distribution, which in fact generalize the Wigner's semicircle law to a larger extent.

Theorem 3.1 *For $p = \omega(\frac{1}{n})$, the empirical spectral distribution (ESD) of the matrix $\frac{1}{\sqrt{n}}M_n$ converges in distribution to the standard semicircle distribution which has a density $\rho_{sc}(x)$ with support on $[-2, 2]$,*

$$\rho_{sc}(x) := \frac{1}{2\pi}\sqrt{4 - x^2}.$$

We first estimate the skew energy $\mathcal{E}_S(G^\sigma(n,p))$ by applying the theorem above but leave the proof of the theorem at the end of this section. Clearly, $\frac{1}{\sqrt{p}}\lambda_1, \frac{1}{\sqrt{p}}\lambda_2, \ldots, \frac{1}{\sqrt{p}}\lambda_n$ and $\frac{1}{\sqrt{np}}\lambda_1, \frac{1}{\sqrt{np}}\lambda_2, \ldots, \frac{1}{\sqrt{np}}\lambda_n$ are the eigenvalues of M_n and $\frac{1}{\sqrt{n}}M_n$, respectively. By Theorem 3.1, we can deduce that

$$\begin{aligned}
\frac{\mathcal{E}_S(G^\sigma(n,p))}{n^{3/2}p^{1/2}} &= \frac{1}{n^{3/2}p^{1/2}}\sum_{i=1}^{n}|\lambda_i| \\
&= \frac{1}{n}\sum_{i=1}^{n}|\frac{1}{\sqrt{np}}\lambda_i| \\
&= \int |x|\, dF_n^{n^{-1/2}M_n}(x) \\
&\xrightarrow{a.s.} \int |x|\, \rho_{sc}(x)\, dx \quad (n \to \infty) \\
&= \frac{1}{2\pi}\int_{-2}^{2}|x|\sqrt{4 - x^2}dx \\
&= \frac{8}{3\pi}.
\end{aligned}$$

Hence, we can immediately conclude that

Theorem 3.2 *For $p = \omega(\frac{1}{n})$, the skew energy $\mathcal{E}_S(G^\sigma(n,p))$ of the random oriented graph $G^\sigma(n,p)$ enjoys a.s. the following equation:*

$$\mathcal{E}_S(G^\sigma(n,p)) = n^{3/2}p^{1/2}\left(\frac{8}{3\pi} + o(1)\right).$$

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1: Let $W_n = \frac{1}{\sqrt{n}}M_n$. To prove that the empirical spectral distribution of W_n converges in distribution to the standard semicircle distribution, it suffices to show that the moments of the empirical spectral distribution converge almost surely to the corresponding moments of the semicircle distribution.

For a positive integer k, the k-th moment of the ESD of the matrix W_n is

$$\int x^k dF_n^{W_n}(x) = \frac{1}{n}\mathbf{E}\left(\mathrm{Trace}(W_n^k)\right),$$

and the k-th moment of the standard semicircle distribution is

$$\int_{-2}^{2} x^k \rho_{sc}(x)dx.$$

Hence, we need to prove for every fixed integer k,

$$\frac{1}{n}\mathbf{E}\left(\mathrm{Trace}(W_n^k)\right) \longrightarrow \int_{-2}^{2} x^k \rho_{sc}(x)dx, \text{ as } n \to \infty.$$

On one hand, we can determine that
for $k = 2m+1$, $\int_{-2}^{2} x^k \rho_{sc}(x)dx = 0$ due to symmetry;
for $k = 2m$,

$$\begin{aligned}
\int_{-2}^{2} x^k \rho_{sc}(x)dx &= \frac{1}{2\pi}\int_{-2}^{2} x^k \sqrt{4-x^2}dx = \frac{1}{\pi}\int_{0}^{2} x^{2m}\sqrt{4-x^2}dx \\
&= \frac{2^{2m+1}}{\pi}\int_0^1 y^{m-1/2}(1-y)^{1/2}dy \text{ (by setting } x = 2\sqrt{y}) \\
&= \frac{2^{2m+1}}{\pi}\cdot\frac{\Gamma(m+1/2)\Gamma(3/2)}{\Gamma(m+2)} = \frac{1}{m+1}\binom{2m}{m}.
\end{aligned}$$

On the other hand, we expand the trace of W_n^k into

$$\begin{aligned}
\frac{1}{n}\mathbf{E}\left(\mathrm{Trace}(W_n^k)\right) &= \frac{1}{n^{1+k/2}}\mathbf{E}\left(\mathrm{Trace}(M_n^k)\right) \\
&= \frac{1}{n^{1+k/2}} \sum_{1\le i_1,\ldots,i_k\le n} \mathbf{E}(m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1}).
\end{aligned} \tag{3.2}$$

Every term in the sum above corresponds to a closed walk of length k in the complete graph of order n. Recall that the matrix M_n satisfies that the entries m_{ij}, $1 \le i < j \le n$, are i.i.d. copies of the random variable ξ, which commits $|\mathbf{E}(\xi^s)| = 0$ if s is odd and $|\mathbf{E}(\xi^s)| = \frac{1}{(\sqrt{p})^{s-2}}$ if s is even. Besides, $m_{ij} = \overline{m_{ji}} = -m_{ji}$. For convenience, we also regard m_{ij} as an edge and m_{ji} the inverse edge of m_{ij}, or vice versa.

When k is odd, each walk in the Sum (3.2) contains such an edge that the total number of times that this edge and its inverse edge appear in this walk is odd. Apparently, by Equ.(3.1) and the independence of the variables, this term is zero. Thus

$$\frac{1}{n}\mathbf{E}\left(\mathrm{Trace}(W_n^k)\right) = 0.$$

When k is even, suppose $k = 2m$ and let t be the number of distinct vertices in a closed walk. All closed walks in the Sum (3.2) can be classified into the following two categories:

Category 1: There exists such an edge in the closed walk that the total number of times that this edge and its inverse edge appear is odd. Similarly, by Equ.(3.1) this term is zero.

Category 2: Each edge in the closed walk satisfies that the total number of times that this edge and its inverse edge appear is even. It is clear that the number of distinct vertices in this walk $t \le m+1$. We then continue to divide those walks into the following two subcategories:

Subcategory 2.1: $t \leq m$. It is clear that the number of such closed walks is at most $n^t \cdot t^k$. Then these terms will contribute

$$\begin{aligned}
&\frac{1}{n^{1+k/2}} \sum_{t=1}^{m} \sum_{|\{i_1,\ldots,i_k\}|=t} \left|\mathbf{E}(m_{i_1i_2}m_{i_2i_3}\cdots m_{i_ki_1})\right| \\
\leq&\frac{1}{n^{1+m}} \sum_{t=1}^{m} n^t \cdot t^k \cdot \left(\frac{1}{\sqrt{p}}\right)^{2m-2(t-1)} \\
\leq&\frac{1}{n^{1+m}} \cdot m \cdot n^m \cdot m^t \cdot \left(\frac{1}{\sqrt{p}}\right)^{2m-2(m-1)} \\
=&\frac{m^{t+1}}{np} = O\left(\frac{1}{np}\right).
\end{aligned}$$

The first inequality is obtained by merging the same edges and their inverse edges together and then employing Equ.(3.1). The second inequality is due to the monotonicity.

Subcategory 2.2: $t = m+1$. In this case, each edge in the closed walk appears only once, and so does its inverse edge. By $\mathbf{E}(\xi\bar{\xi}) = -\mathbf{E}(\xi^2) = 1$ and the independence of the variables, this term is 1. And the number of such closed walk is given by the following lemma.

Lemma 3.3 *[2] The number of the closed walks of length $2m$ which satisfy that each edge and its inverse edge in the closed walk both appear once is $\frac{1}{m+1}\binom{2m}{m}$.*

From the above discussion, it follows that

$$\frac{1}{n}\mathbf{E}\left(\mathrm{Trace}(W_n^k)\right) = \begin{cases} 0 & \text{if } k = 2m+1; \\ \frac{1}{m+1}\binom{2m}{m} + O\left(\frac{1}{np}\right) & \text{if } k = 2m, \end{cases}$$

which implies that if $p = \omega(\frac{1}{n})$, then

$$\frac{1}{n}\mathbf{E}\left(\mathrm{Trace}(W_n^k)\right) \ \rightarrow\ \int_{-2}^{2} x^k \rho_{sc}(x)dx, \text{ as } n \rightarrow \infty.$$

The proof is thus completed. ∎

4 The skew energy of $G_{n,d}^{\sigma}$

In this section, we consider the skew energy of random regular oriented graphs. We first recall [3] the definition of a random regular graph $G_{n,d}$, where $d = d(n)$ denotes the degree. $G_{n,d}$ is a random graph chosen uniformly from the set of all simple d-regular graphs on n vertices. A *random regular oriented graph*, denoted by $G_{n,d}^{\sigma}$, is obtained by orienting each edge of the random regular graph $G_{n,d}$, randomly and independently, with probability 1/2. Let A_n

be the adjacency matrix of $G_{n,d}$ and R_n be the skew-adjacency matrix of $G^\sigma_{n,d}$. The estimates of the skew energy of $G^\sigma_{n,d}$ are different in the cases of d fixed and $d \to \infty$. Therefore, we shall discuss these two cases separately.

4.1 The case that $d \geq 2$ is a fixed integer

In this subsection, we estimate the skew energy of $G^\sigma_{n,d}$, where $d \geq 2$ is a fixed integer. We first recall the fact about the limiting spectral distribution of the random regular graph $G_{n,d}$ with d fixed (which means the limiting spectral distribution of the adjacent matrix A_n), which was derived by McKay [13].

Lemma 4.1 *[13] Let $G_{n,d}$ be a random regular graph with the adjacency matrix A_n. If the degree d is a fixed integer and $d \geq 2$, then the empirical spectral distribution $F_n^{A_n}$ approaches the distribution $F(x)$ whose density function is*

$$\rho_d = \begin{cases} \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}, & \text{if } |x| \leq 2\sqrt{d-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 4.1. McKay [13] used the moment method to prove the lemma above, i.e., he proved that for each k, the k-th moment of the ESD of the matrix A_n converges to the k-th moment of the distribution $F(x)$,

$$\int x^k dF_n^{A_n}(x) = \frac{1}{n}\mathbf{E}\left(\text{Trace}(A_n^k)\right) \longrightarrow \int x^k \rho_d(x)dx, \text{ as } n \to \infty.$$

Note that $\text{Trace}(A_n^k)$ is the number of closed walks of length k in A_n. When d is fixed, the graph $G_{n,d}$ is almost surely a locally d-regular tree.

Now we consider the random regular oriented graph $G^\sigma_{n,d}$. Set $T_n = (-i)R_n = [t_{ij}]$. For a fixed k,the limit of the k-th moment of the ESD of T_n is $m_k = \lim\limits_{n\to\infty} \frac{1}{n}\mathbf{E}\left(\text{Trace}(T_n^k)\right)$. We note that when d is fixed and $n \to \infty$, the underlying graph is almost surely a locally d-regular tree. If one oriented edge appears in a closed walk of length k, then its inverse oriented edge appears with the same number of times. We can get that m_k is equal to the number of closed walks of length k in a d-regular tree starting at the root. Combining with Lemma 4.1 we conclude the following theorem.

Theorem 4.2 *Let $G^\sigma_{n,d}$ be a random regular oriented graph with the adjacency matrix R_n, and let $T_n = (-i)R_n = [t_{ij}]$. If the degree d is a fixed integer and $d \geq 2$, then the empirical spectral distribution $F_n^{T_n}$ approaches the distribution $F(x)$ which has the density function ρ_d with support on $\left[-2\sqrt{d-1}, 2\sqrt{d-1}\,\right]$,*

$$\rho_d = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}.$$

We now turn to the estimate of the skew energy $\mathcal{E}_S(G^{\sigma}_{n,d})$. Note that $\mathcal{E}_S(G^{\sigma}_{n,d})$ also equals the sum of the absolute values of all the eigenvalues of R_n. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of R_n. By Theorem 4.2, we can deduce that

$$\begin{aligned}
\frac{\mathcal{E}_S(G^{\sigma}_{n,d})}{n} &= \frac{1}{n}\sum_{i=1}^{n}|\lambda_i| \\
&= \int |x|\, dF_n^{R_n}(x) \\
&\xrightarrow{a.s.} \int |x|\, \rho_d(x)\, dx \quad (n \to \infty) \\
&= 2\int_0^{2\sqrt{d-1}} x\, \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}\, dx \\
&= \frac{2d\sqrt{d-1}}{\pi} - \frac{d(d-2)}{\pi}\cdot \arctan\frac{2\sqrt{d-1}}{d-2}.
\end{aligned}$$

To summarize, we can obtain the following theorem.

Theorem 4.3 *For any fixed integer $d \geq 2$, the skew energy $\mathcal{E}_S(G^{\sigma}_{n,d})$ of the random regular oriented graph $G^{\sigma}_{n,d}$ enjoys a.s. the following equations:*

$$\mathcal{E}_S(G^{\sigma}_{n,d}) = n\left(\frac{2d\sqrt{d-1}}{\pi} - \frac{d(d-2)}{\pi}\cdot \arctan\frac{2\sqrt{d-1}}{d-2} + o(1)\right).$$

In particular, when $d = 2$, $\mathcal{E}_S(G^{\sigma}_{n,d}) = n\,(4/\pi + o(1))$.

4.2 The case that $d \to \infty$

The estimate of the skew energy of $G^{\sigma}_{n,d}$ with $d \to \infty$ depends on the following key lemmas.

Lemma 4.4 *[17] If $np \to \infty$, then the random graph $G(n,p)$ is np-regular with probability at least* $\exp\left(-O(n(np)^{1/2})\right)$.

Next, we consider random oriented graphs. By the definitions of a random oriented graph and a random regular oriented graph, we can generalize the lemma above into a result for random oriented graphs as follows.

Lemma 4.5 *If $np \to \infty$, then the random oriented graph $G^{\sigma}(n,p)$ is np-regular with probability at least* $\exp\left(-O(n(np)^{1/2})\right)$.

Lemma 4.6 *[17] Let M be an $n \times n$ Hermitian random matrix whose off-diagonal entries ξ_{ij} are i.i.d. random variables with mean zero, unit variance and $|\xi_{ij}| < K$ for some common constant K. Fix $\delta > 0$ and assume that the fourth moment $M_4 := \sup_{i,j} \mathbf{E}(|\xi_{ij}|^4) = o(n)$. Then for any interval $I \subset [-2,2]$ whose length is at least $\Omega(\delta^{-2/3}(M_4/n)^{1/3})$, there is a constant c*

such that the number N_I of the eigenvalues of $\frac{1}{\sqrt{n}}M$ which belong to I satisfies the following concentration inequality

$$P\left(\left|N_I - n\int_I \rho_{sc}(t)dt\right| > \delta n \int_I \rho_{sc}(t)dt\right) \le 4\exp\left(-c\frac{\delta^4 n^2 |I|^5}{K^2}\right).$$

Consider the random oriented graph $G^\sigma(n,p)$ with $np \to \infty$ as $n \to \infty$ and the skew-adjacency matrix S_n. Recall that $M_n = \frac{-i}{\sqrt{p}}S_n$. For an interval I let N'_I be the number of eigenvalues of M_n in I. Apparently, M_n satisfies the condition of Lemma 4.6 ($M := M_n$, $K := 1/\sqrt{p}$). Thus, we can immediately obtain the following lemma.

Lemma 4.7 *For any interval $I \subset [-2,2]$ with length at least $(\frac{\log(np)}{\delta^4(np)^{1/2}})^{1/5}$, we have*

$$\left|N'_I - n\int_I \rho_{sc}(x)dx\right| > \delta n \int_I \rho_{sc}(x)dx$$

with probability at most $\exp(-cn(np)^{1/2}\log(np))$.

By Lemmas 4.5 and 4.7, the probability that N'_I fails to be close to the expected value in the model $G^\sigma(n,p)$ is much smaller than the probability that $G^\sigma(n,p)$ is np-regular. Thus, the probability that N'_I fails to be close to the expected value in the model $G^\sigma_{n,d}$ where $d = np$ is the ratio of the two former probabilities, which is $O(\exp(-cn\sqrt{np}\log np))$ for some small positive constant c. Recall that R_n is the skew-adjacency matrix of $G^\sigma_{n,d}$. Set $L_n = \frac{-i}{\sqrt{d/n}}R_n$ and let N''_I be the number of eigenvalues of L_n in I. Thus we can conclude that

Theorem 4.8 *(Concentration for ESD of $G^\sigma_{n,d}$) Let $\delta > 0$ and consider the random regular oriented graph $G^\sigma_{n,d}$. If d tends to ∞ as $n \to \infty$, then for any interval $I \subset [-2,2]$ with length at least $\delta^{-4/5}d^{-1/10}\log^{1/5} d$, we have*

$$\left|N''_I - n\int_I \rho_{sc}(x)dx\right| < \delta n \int_I \rho_{sc}(x)dx$$

with probability at least $1 - O(\exp(-cn\sqrt{d}\log(d)))$.

Theorem 4.8 immediately implies a result as follows.

Theorem 4.9 *If $d \to \infty$, then the ESD of $n^{-1/2}L_n$ converges to the standard semicircle distribution.*

Now we are ready to estimate the skew energy of the random regular oriented graph $G^\sigma_{n,d}$. Suppose that $i\lambda_1, i\lambda_2, \ldots, i\lambda_n$ are the eigenvalues of R_n. Then, $d^{-1/2}\lambda_1, d^{-1/2}\lambda_2, \ldots,$ $d^{-1/2}\lambda_n$ are the all eigenvalues of $n^{-1/2}L_n$. By Theorem 4.9, we can deduce that

$$
\begin{aligned}
\frac{\mathcal{E}_S(G^\sigma_{n,d})}{nd^{1/2}} &= \frac{1}{nd^{1/2}}\sum_{i=1}^{n}|\lambda_i| \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{d}}|\lambda_i| \\
&\xrightarrow{a.s.} \int |x|\,\rho_{sc}\,dx \quad (n\to\infty) \\
&= \frac{1}{2\pi}\int_{-2}^{2}|x|\sqrt{4-x^2}dx \\
&= \frac{8}{3\pi}.
\end{aligned}
$$

Therefore, the skew energy $\mathcal{E}_S(G^\sigma_{n,d})$ can be formulated as

$$\mathcal{E}_S(G^\sigma_{n,d}) \;=\; nd^{1/2}\left(\frac{8}{3\pi}+o(1)\right).$$

We can thus immediately obtain the following theorem.

Theorem 4.10 *For $d = d(n) \to \infty$, the skew energy $\mathcal{E}_S(G^\sigma_{n,d})$ of the random oriented graph $G^\sigma_{n,d}$ enjoys a.s. the following equation:*

$$\mathcal{E}_S(G^\sigma_{n,d}) \;=\; nd^{1/2}\left(\frac{8}{3\pi}+o(1)\right).$$

References

[1] C. Adiga, R. Balakrishnan, W. So, The shew energy of a digraph, Linear Algebra Appl. 432(2010), 1825–1835.

[2] Z.D. Bai, J.W. Silverstein, Spectral Analysis of Large Dimensional Random Matrices, Second Edition, Springer Series in Statistics, 2010.

[3] B. Bollobás, Random Graphs, Cambridge University Press, 2001.

[4] W. Du, X. Li, Y. Li, The Laplacian energy of random graphs, J. Math. Anal. Appl. 368(2010), 311–319.

[5] W. Du, X. Li, Y. Li, Various energies of random graphs, MATCH Commun. Math. Comput. Chem. 64(2010), 251–260.

[6] W. Du, X. Li, Y. Li, The energy of random graphs, Linear Algebra Appl. 435(2011), 2334–2346.

[7] S. Gong, G. Xu, 3-Regular digraphs with optimum skew energy, Linear Algebra Appl. 436(2012), 465–471.

[8] I. Gutman, The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz. Graz, 103(1978), 1–22.

[9] I. Gutman, X. Li, J. Zhang, Graph Energy, in: M. Dehmer, F. Emmert-Streib (Eds.), Analysis of Complex Network: From Biology to Linguistics, Wiley-VCH Verlag, Weinheim, 2009, 145–174.

[10] Y. Hou, T. Lei, Characteristic polynomials of skew-adjacency matrices of oriented graphs, Electron. J. Combin. 18(2011), 156–167.

[11] Y. Hou, X. Shen, C. Zhang, Oriented unicyclic graphs with extremal skew energy, Available at http://arxiv.org/abs/1108.6229.

[12] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.

[13] B.D. McKay, The expected eigenvalue distribution of a large regular graph, Linear Algebra Appl. 40(1981), 203–216.

[14] M.L. Mehta, Random Matrices, 2nd ed. Academic Press, 1991.

[15] B. Shader, W. So, Skew spectra of oriented graphs, Electron. J. Combin. 16(2009), #N32.

[16] X. Shen, Y. Hou, C. Zhang, Bicyclic digraphs with extremal skew energy, Electron. J. Linear Algebra 23(2012), 340–355.

[17] L.V. Tran, V.H. Vu, K. Wang, Sparse random graph: eigenvalues and eigenvectors, Random Struct. Alg. 42(2013), 110–134.

[18] E.P. Wigner, Characteristic vectors of bordered matrices with infinite dimmensions, Ann. Math. 62(1955), 548–564.

[19] E.P. Wigner, On the distribution of the roots of certain symmetric matrices, Ann. Math. 67(1958), 325–327.